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THE LENGTH OF THE LUNAR MONTH

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1. *Introduction*

Many old and modern cultures use lunar calendars. That is, the lunar month is based on the time interval from the moon passing from one phase around a complete cycle to return to the same phase. In the Babylonian calendar, the ancient Jewish calendar, and Islamic calendar, the start of the lunar month was the time that the thin crescent moon was first sighted after new moon. In the modern Jewish calendar, the start of the month is associated with a calculation of the instant of new moon. The Christian date of Easter is based on an idealized calculation of the full moon time. Lunar or lunisolar calendars were and are also used by Indian, American Indian, Chinese, paleolithic, and neolithic cultures.

Many studies of lunar or lunisolar calendars depend on a knowledge of the length of the lunar month. For example, at the recent Third International Conference on Archaeoastronomy (Oxford 3) in St Andrews, Scotland, several speakers presented various assumptions and pleas for more information concerning the length of the month. One typical application was an attempt to associate the twenty-eight lunar lodges used widely throughout Asia with the number of days that the moon is visible. Another application used by several speakers was to present tallies of numbers preserved on bones (with dates ranging from the paleolithic to *c.* 1800 A.D.) and associate the derived numbers with observations of the lunar cycle. Finally, the application to Islamic calendrics was discussed.

The traditional answer to questions of the length of the month is that the mean synodic period of the moon is 29.53 days, so that a month composed of a whole number of days will be either 29 or 30 days long. But reality is not that simple. First, the actual time from, say, new moon to new moon can vary from 29.2679 to 29.8376 days.[1] Second, there are two possible points of view, with the geocentric phase being a global phenomenon of theoretical existence and the topocentric phase being a local phenomenon related to actual observation. Third, for those calendars based on observation of the crescent, the complex variations of the moon's position increases the variance of the month's duration. Fourth, for those calendars based on observation, the presence of clouds can also prolong some months.

Aside from this traditional answer, the only previous detailed examination of the length of the lunar month is by Huber.[2] He has calculated the month length with the crescent visibility criterion of Neugebauer[3] for a large number of months. Huber finds that the months' lengths are 29 days (46.94%), 30 days (53.00%), and 31 days (0.06%), with some long periodicities. However,

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TABLE 1. Lunar months for Meridian Mississippi for 1930–31 (cloudfree conditions).

Year	Old Moon, No Clouds	Duration of Dark Period	Young Moon, No Clouds	Duration of Month
1930	Jan 27	3.5	Jan 30	30
1930	Feb 26	4.5	Mar 1	30
1930	Mar 27	4.5	Mar 31	29
1930	Apr 26	3.5	Apr 29	30
1930	May 26	3.5	May 29	29
1930	Jun 25	2.5	Jun 27	29
1930	Jul 24	2.5	Jul 26	30
1930	Aug 22	3.5	Aug 25	29
1930	Sep 21	2.5	Sep 23	30
1930	Oct 20	3.5	Oct 23	29
1930	Nov 18	3.5	Nov 21	30
1930	Dec 18	3.5	Dec 21	29
1931	Jan 16	3.5	Jan 19	30
1931	Feb 15	3.5	Feb 18	30
1931	Mar 17	3.5	Mar 20	30
1931	Apr 15	4.5	Apr 19	29
1931	May 15	3.5	May 18	30
1931	Jun 14	3.5	Jun 17	29
1931	Jul 14	2.5	Jul 16	30
1931	Aug 12	3.5	Aug 15	29
1931	Sep 10	3.5	Sep 13	29
1931	Oct 10	2.5	Oct 12	30
1931	Nov 8	3.5	Nov 11	29
1931	Dec 7	3.5	Dec 10	29
1932	Jan 6	2.5	Jan 8	

Neugebauer's criterion has the serious fault that the seasonal variations in the extinction coefficients are not taken into account.

In this paper, I will quantify the three effects of the length of the lunar month for three different locations for six years. I will present histograms of the length of the lunar month for the conditions of (1) with no clouds, (2) with clouds, and (3) with clouds and a rule that a month can not be longer than 30 days. I will also discuss the length of the time when the moon is not visible, the degree by which months alternate between 29 and 30 days, and the relations between observations at nearby sites. In addition, I will discuss the accuracy by which the full moon and quarter moon phases can be measured.

2. *Cloudless Conditions*

The task of predicting the visibility of the thin lunar crescent is an old one, dating at least back to the Babylonians. Before modern times, the typical prediction algorithm merely stated that the moon would be visible if its distance from the sun (in one or more coordinates) was greater than some threshold distance. The best of these criteria is that of Fotheringham.[4] All these algorithms have the great drawback that the entire world is implicitly assumed to have the exact same atmospheric conditions as those used to construct the

TABLE 2. Summary of statistics for cloud free conditions.

Site:	Meridian	Green Bay	Tucson
Month is 29 days	49%	49%	49%
Month is 30 days	51%	51%	51%
Dark period is 2.5 days	36%	27%	49%
Dark period is 3.5 days	56%	46%	48%
Dark period is 4.5 days	8%	27%	3%
Alternating months (1st order)	68%	56%	72%
Alternating months (2nd order)	59%	51%	59%
Alternating months (3rd order)	48%	46%	55%

criterion. This is a poor approximation, as the crescent visibility from the swamps of Louisiana is greatly more difficult than from the crystal clear skies of the American Southwest.

To overcome these and other problems, I have developed a prediction algorithm based on the astronomy, meteorology, and physiology of the detection process.[5] The method employed was to model mathematically every physical process that affects the moonlight from its reflection to its detection. The processes modelled include the microscopic and macroscopic shadowing on the moon, the lunar albedo, the relative positions of the sun, moon, and horizon, the variation of the extinction coefficient as a function of the date, latitude, relative humidity, time of year, longitude, and altitude, the variation of the optical pathlength in the atmosphere for each extinction component, the atmospheric refraction, the brightness of the twilight sky, and the detection probability of the human eye for the calculated conditions. The entire algorithm has been published as a computer program.[6]

It might be useful to reiterate typical and extreme results from my algorithm. For typical conditions, the age of the moon when first sighted from a location can be as young as 20 hours or as old as several days. This age will depend on the time of year, as during the spring the moon is directly over the sun at sunset so that the distance the moon must travel from conjunction is minimal, while during the autumn the moon is usually far to one side of the sun at sunset so that even though the moon is far from the sun (with a large age) the moon will set quickly after the sun and be invisible. The age of first visibility will also depend on the distance of the moon, since at perigee the moon's angular motion is greater than at apogee so that the moon will achieve a critical separation from the sun at a younger age. The age of first visibility will also depend on the observer's longitude, in that one observer might spot a marginally visible crescent yet observers to the east will first see a moon that is one day older. Danjon[7] and Schaefer[8] demonstrate observationally and theoretically that the crescent is invisible to ground-based visual observers when it is within 7° of the sun. However, for real conditions this limit is never achieved. The youngest moon ever (reliably) seen is 15.4 hours by J. Schmidt[9] with the unaided eye and 13.4 hours by R. Victor for an observation with optical aid.[10]

My algorithm has undergone tests against extensive sets of observations. A

total of 201 observations have been collected[11] from the astronomical literature and the algorithm is found to be over 2.3 times more accurate than any other published algorithm. An additional 51 observations have been recently collected[12] from the astronomical literature with the same conclusion. Several Moonwatches have been organized[13] where over 1500 observers throughout North America reported whether the crescent was sighted or not. These observations all yield the strong conclusion that the algorithm is a good predictor of the observations.[14]

Therefore, it is with confidence that I use the program to calculate the visibility of the crescent. These calculations were performed for three sites selected for the high quality of the extinction data available from Reddy.[15] The three sites were Meridian Mississippi, Green Bay Wisconsin, and Tucson Arizona. The sites were chosen to represent a northern and a southern temperate latitude as well as good and poor atmospheric clarity. The calculations were performed for three periods each with two years duration, namely 1930–31, 1935–36, and 1940–41. The distribution was to test for possible changes throughout the 18.6 year period of the moon's nodal motion. The early dates were chosen so as to avoid the heavy modern pollution in the summer for the American East[16] which is automatically taken into account by the program.

The results of these calculations for Meridian for 1930–31 are presented in Table 1 as an illustration, while the information for the remaining dates and sites is not given to save space. This list of dates can be used to construct the lengths of lunar months on the presumption that clouds do not obscure the crescent (for examples, see the last column of Table 1). Various relevant statistical information gleaned from my tables is presented in Table 2.

For me, the most surprising result is that the length of the observed month is always 29 or 30 days long and is never 28 or 31 days long. The reason for my expectations of long or short months was that the moon can be first visible as young as 20 hours (in spring) and as old as several days (in the autumn), so that a chance juxtaposition of these conditions could lead to long or short months. The fallacy in my expectations is that the conditions change on a time scale of much longer than a month; thus, adjacent months have similar lunar distances, similar lunar ecliptic latitudes, and similar azimuths relative to the sun. So even though the visibility changes greatly with these parameters, the effects for adjacent new moons are similar and hence the month length will not be much affected.

Another result is the almost perfect division between months of 29 and 30 days. This is a necessary result in the long run as the mean synodic month is almost exactly halfway between 29 and 30 days. One implication is that a calendar that alternates months of 29 and 30 days will not have large long-term errors.

The months have a non-random tendency to alternate their lengths between 29 and 30 days. This tendency can be quantified by the first order probability (P) that any given month length will be followed by a month with a different length. For the three sites, this probability ranges from 56% to 72%, so the months do tend to alternate at roughly two-to-one odds. A second order

statistic can be defined as the probability that two months separated by one month will have the same length. A third order statistic can be defined as the probability that two months separated by two months will have different lengths. The second and third order statistics are much closer to chance (50%) than is the first order statistic, which implies that the alternating does not have long-term order. If the alternation of adjacent month lengths were independent, then the second and third order probabilities should be $P^2+(1-P)^2$ and $P^3+3P(1-P)^2$ respectively, and this is just as observed. Therefore, I conclude that there is a distinct tendency for the months to alternate in length, but that no long-term correlations exist.

The sites were spread out widely across the North American continent, yet all three sites first saw the crescent on the same day 58% of the time. This shows that, for the much smaller regions relevant to early cultures or to many modern nations, it is reasonable to apply one prediction for the entire nation. Of the cases without unanimity, Meridian, Green Bay, and Tucson were the exception 7%, 19%, and 16% of the time respectively.

The length of the dark period ranges from roughly 2.5 days to 4.5 days. That is, a 2.5-day interval counts from the morning of the first day to the evening of the third day and includes two midnights. The dates of old moon sightings and the corresponding length of the dark periods during 1930 and 1931 are tabulated for Meridian Mississippi in Table 1. On no occasion did a dark interval of 1.5 days occur, although one lunation as viewed from Tucson came close. (From Tucson, the old crescent was easily visible on the morning of 28 November 1940 and was marginally invisible ($R=-0.2\pm0.4$, see Schaefer[17]) on the evening of 29 November 1940.) A dark interval of roughly 1.5 days (actually the record is 1.599 days[18]) is possible[19] although difficult and requires either extremely clear skies or optical assistance. The myths that Kepler and Amerigo Vespucci both saw the young and old Moon on the same day[20] can only be false. With a dark period of 2.5 to 4.5 days and a synodic month of 29.5 days, the length of time that the moon is visible during a lunation is 25 to 27 days.

3. *Cloudy Conditions*

The statistics quoted in the previous section are relevant for observations made when the critical nights do not have clouds low on the horizon to obstruct the view. For many applications, this is the relevant case, as the culture may have selected values that were obtained when the weather was good. However, for various applications, the reality of clouds is relevant. For example, when paleolithic man constructed his tallies, he did not wait for a year free of clouds on the critical dates. And Islamic calendars must still be constructed without ignoring the imperfect months.

I have collected statistics on the rate of significant clouds low on the dawn and dusk twilight sky[21] as part of a program of measuring the heliacal rising of stars. These statistics are for 375 mornings and 410 evenings at ten sites around the world. These statistics are directly applicable to the case of looking for the crescent in twilight.

TABLE 3. Lunar months for Meridian Mississippi for 1930–31 (cloudy conditions).

Year	Young Moon with Clouds		Young Moon with ⩽30 Day Rule	
	Date of First Visibility	Length of Month	Date of Start of Month	Length of Month
1930	Jan 30	34	Jan 30	30
1930	Mar 4	28	Mar 1	30
1930	Apr 1	31	Mar 31	30
1930	May 2	28	Apr 30	30
1930	May 30	29	May 30	29
1930	Jun 28	29	Jun 28	29
1930	Jul 26	31	Jul 26	30
1930	Aug 26	32	Aug 25	30
1930	Sep 27	28	Sep 24	30
1930	Oct 25	28	Oct 24	29
1930	Nov 22	30	Nov 22	30
1930	Dec 22	30	Dec 22	30
1931	Jan 21	28	Jan 21	28
1931	Feb 18	31	Feb 18	30
1931	Mar 21	29	Mar 20	30
1931	Apr 19	35	Apr 19	30
1931	May 24	26	May 19	30
1931	Jun 19	36	Jun 18	30
1931	Jul 25	24	Jul 18	30
1931	Aug 18	29	Aug 17	30
1931	Sep 16	27	Sep 16	27
1931	Oct 13	32	Oct 13	30
1931	Nov 14	33	Nov 12	30
1931	Dec 17	24	Dec 12	29
1932	Jan 10		Jan 10	

I find that an excellent site (like Tucson) has roughly half of the nights clear enough to be used, while an average site (like Meridian or Green Bay) has a third of the nights clear enough of clouds to be useable. The fraction of clear mornings is somewhat larger than the fraction of clear evenings. There is no indication in the data of seasonal changes, although such effects are known to exist, for example as the 'monsoon season' in Arizona.

The weather is strongly correlated from one day to the next. That is, both cloudy and clear weather last for durations much longer than one day. This is a consequence of the several-day travel time for a weather system to pass through a region. My cloudiness statistics show that any site has a roughly 75% probability that any given cloudy night will be immediately followed by another cloudy night. For an excellent site, any given clear night has a roughly 25% probability of being followed by a cloudy night. For an average site, any given clear night has a roughly 50% probability of being followed by a cloudy night.

With this information, it is possible to simulate the weather conditions for any of the three sites in this study. In detail, I choose the weather for the first night of a lunation by selecting a random number from zero to one and then check if it is larger or smaller than the cloudiness probability. If the sky is

TABLE 4. Summary of month lengths for cloudy conditions.

Month Length	Site Meridian	Green Bay	Tucson
15 days	1	0	0
16	0	1	0
17	0	0	0
18	0	0	0
19	0	1	0
20	1	1	1
21	1	1	0
22	1	2	2
23	0	2	0
24	3	1	1
25	3	5	1
26	6	1	5
27	2	5	2
28	8	5	2
29	14	11	26
30	8	10	20
31	8	7	4
32	3	6	3
33	3	4	2
34	4	2	2
35	3	3	0
36	3	2	0
37	1	0	1
38	1	1	1
39	0	0	0
40	0	0	0
41	0	1	0
42	0	0	0
43	0	1	0
44	0	0	0
45	0	1	0

chosen to be clear, then the moon should be seen. If the first night was cloudy, I select a random number from zero to one and then check to see if it is larger or smaller than the probability that a clear night will follow a cloudy night. This procedure is repeated until a clear night is obtained, on which the moon is taken to be visible. Examples of the simulated delays caused by clouds are given in Table 3 for Meridian Mississippi from 1930 to 1931 (*cf.* Table 1).

If a lunar calendar were based purely on the dates for which the moon is first seen, then the length of the month can vary widely. The extreme cases in this study are 15 and 45 days. The distribution for the three sites is shown in Table 4. The rms scatter of the length is 4.0, 5.0, and 2.8 days for Meridian, Green Bay, and Tucson respectively. If the cloudiness fraction were zero, then the rms scatter would be 0.5 days.

It is possible for a culture to have some mechanism for avoiding the extremes caused by clouds. One such mechanism for the Islamic calendar has justifica-

TABLE 5. Summary of month lengths for cloudy conditions with a 30-day maximum length.

	Site		
Month Length	Meridian	Green Bay	Tucson
26 days	1%	0%	0%
27	4%	3%	1%
28	3%	6%	0%
29	24%	22%	43%
30	68%	69%	56%

tion[22] in the Sahih Muslim chapter no. 2378. For this mechanism, the months are started on the basis of actual observation, unless it is cloudy, in which case the new month is started 30 days after the previous month's start. With this system, I can calculate the lengths of the month as illustrated in Table 3. The resulting distribution of month lengths for the three sites is presented in Table 5. This Islamic method for regulating the calendar turns out to be remarkably good at avoiding discrepant months. That is, it is rare for a month not to have either 29 or 30 days, and these months are always short by only from one to three days.

4. *Full Moon*

The lunar month need not be defined as from new moon to new moon. Instead, it could be defined as from full moon to full moon. Times of the full moon are used in various calendric systems, perhaps the best known being the timing of Easter in the Christian calendar. Festivals in many cultures (for example, the Zuni Shalako ceremony[23]) are tied in to the time of full moon. Many calendars in use in India and Sri Lanka have the months starting at full moon.[24]

The modern astronomical definition of full moon is the instant when the geocentric ecliptic longitudes of the sun and moon differ by exactly 180°. As such, the astronomical full moon is not directly observable. Nevertheless, other definitions (involving the terminator shape, shading across the surface, the altitude of the moon at sunset, and the relative azimuth of the moon at sunset) are available for which direct observations can be made. These more practical definitions would presumably have been used by early cultures. How accurately can the instant of full moon be measured by primitive observers?

The accuracy for timing the full moon can be estimated by simple observations. I have made 32 estimates using only simple methods involving the naked eye. The rigour of my observations must be characterized as 'casual', and the observing time was always about two minutes, with regard only for that night's judgement. These methods include a judgement of whether the moon was out of round, a judgement of the relative darkness of the two limbs, a comparison of the altitude of the sun and moon, and a comparison of the moon's azimuth with respect to my shadow's azimuth. I find that, on any one given night of observation within a day or two of the full moon, the time of full moon can be reliably estimated with an average error of 8.3 hours. Given that two minutes of casual watching on one night can establish full moon this

TABLE 6. Surface brightness along the luminance equator of the near full moon.

	Fraction Across Disk					
Phase Angle	$0.9R_{moon}$	$0.7R_{moon}$	$0.5R_{moon}$	$-0.5R_{moon}$	$-0.7R_{moon}$	$-0.9R_{moon}$
2.5°	8.10	7.52	7.30	7.27	7.47	7.99
5.0°	8.57	7.94	7.70	7.65	7.84	8.36
7.5°	8.80	8.10	7.85	7.76	7.95	8.45
10.0°	8.98	8.20	7.93	7.82	8.00	8.50
12.5°	9.17	8.29	8.00	7.85	8.03	8.52

accurately, a series of careful observations over several nights can undoubtedly establish the time to a few hours.

The question of full moon timing can also be approached theoretically for each of the above criteria. First, to a reasonable approximation, the moon will be visibly out of round when the terminator is significantly inside the limb. The resolution of the average human eye is 42″ for typical lunar surface brightnesses.[25] Therefore, when the moon is within 17° of the antisun position, the moon will appear circular. A somewhat more sophisticated analysis (where it is assumed that the observed edge of the moon is fitted to a circle in a least squares sense) suggests a 12° limit for the moon appearing circular.

When the moon is near full, one side will appear somewhat darker than the other, and this will provide a clue to an observer as to the time of full moon. I have calculated the brightness of the near full moon with the complex equations of Hapke[26] and the photometric parameters of Helfenstein and Veverka.[27] These results are presented in Table 6 as surface brightnesses (in magnitudes per unit solid angle) for six locations along the luminance equator for five phase angles from 2.5° to 12.5°. Each location is described by its distance from the centre in units of moon radii, such that positive values are near to the terminator. The threshold level for the detection of brightness differences is difficult to know, especially with the albedo variations across the lunar disk. A reasonable estimate would be 0.3 magnitudes, as appropriate for the comparison of areas that do not share a sharp boundary. Therefore, the moon will appear to have symmetrical shading when it is within roughly 7° of the antisolar position.

The time of full moon is when the moon is closest to the antisolar position. In practice, this position on the sky can be identified to within a degree or so by looking in the direction of the observer's shadow at the time of sunset or sunrise. There will be a one degree bias caused by the refraction on the horizon for both the sun and moon image. This bias will be opposite in the morning when compared to the evening. If observations are carried out shortly before or after sunset, then it is still easy to judge the antisolar position to within a few degrees. The relative position of the moon and the antisun then allows for a reasonable estimate of the time of closest approach since the direction and speed of the moon's motion is well known to any regular skywatcher. The moon's mean sidereal motion is 0.55° per hour, so that a position accurate to several degrees will yield a time of full moon to within roughly a third of a day.

The various observational cues are expected to yield full moon times to

significantly better than a day. With multiple cues and multiple measurements, it should be possible to have an accuracy of well under half a day. Thus, both theory and observation suggest that a half-day accuracy is easy to get, and several hour accuracy is possible for the naked eye with no equipment.

5. *Quarter Moon*

What is the accuracy by which a naked eye observer can estimate the time of the quarter moon? To a reasonable approximation, the moon will appear as a quarter moon if its terminator deviates by less than 42″ from a straight line. Simple trigonometry shows that the terminator will be indistinguishable from a straight line if the moon is within 2.6° of quadrature. Since the moon moves 0.55° per hour on average, the time of the quarter phase should be observable with an accuracy of roughly 5 hours. Once again, multiple observations should be able to achieve better accuracy.

I have made 28 observational estimates of the time of the quarter phase. My technique was crude, in that I used the unaided eye (no straight edge was used) to look at the moon for roughly one minute. The size of any deviations from a straight terminator were then translated to a time from the quarter phase purely by my memory of the night-to-night changes in the terminator shape. I find that I had an average error of 5.5 hours with a maximum error of 21 hours. This observational accuracy is in agreement with the theoretical accuracy.

Aristarchus of Samos (*c.* 310–230 B.C.) derived the relative distance of the sun and moon from the angle between the sun and moon when the moon appeared at exactly half phase. He took the sun/moon angle to be 87° at first quarter, so that the sun would be 19 times farther than the moon. The actual angle is typically 89.85°, a value that cannot be distinguished from 90° by naked-eye observers.

The time of the quarter phase can be determined with a higher accuracy than that of any other phase. That is, the accuracy for the quarter moon phase is roughly 5 hours, while the accuracy for the full moon phase is roughly 10 hours and the accuracy of the first visibility of the crescent phase is quantized to 24 hours. The reason is that the terminator moves across the lunar disk fastest at the quarter phase, so that differences are noticeable in less time. As such, the greatest calendrical accuracy can be achieved by a calendar based on the quarter moon. However, if a culture desires certainty (as opposed to accuracy), then a calendar based on the first visibility of the lunar crescent would be best since the quantization of sighting times enforces uniformity. Since the majority of lunar calendars are based on the crescent phase, I conclude that for most cultures certainty in calendrics is more important than astronomical accuracy.

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